Modus Ponens Tollens for RU-implications

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Introduction

Motivation

- Fuzzy implication functions have been extensively studied in the last decades mainly from two perspectives: from the theoretical point of view and from its possible applications.

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- Fuzzy implication functions have been extensively studied in the last decades mainly from two perspectives: from the theoretical point of view and from its possible applications.
- The study of additional properties is useful to obtain feasible and more adequate fuzzy implication functions in the applications.
- Two of such additional properties are the (generalized) Modus Ponens and Modus Tollens. These properties are of paramount importance in approximate reasoning.

Motivation

- The (generalized) Modus Ponens and Modus Tollens are usually expressed by the following two functional inequalities:

$$T(x, I(x, y)) \leq y, \text{ for all } x, y \in [0, 1],$$

$$T(N(y), I(x, y)) \leq N(x), \text{ for all } x, y \in [0, 1],$$

  where T is a t-norm, I is a fuzzy implication function and N is a fuzzy negation.

Motivation

- These properties have been studied in the literature for the most usual families of fuzzy implication functions such as (S, N), R, QL and D-implications derived from t-norms and t-conorms.

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- Even recently, a whole new line of research has been proposed in which the t-norm T is generalized to a more general conjunction such as a conjunctive uninorm or an overlap function, leading to the so-called U-Modus Ponens or O-Modus Ponens.

Motivation

- Although functional inequalities of Modus Ponens and Modus Tollens have quite similar expressions, it is well known that both properties are not equivalent.

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- Thus, the simultaneous fulfillment of both properties was studied for the first time for some restricted classes of (S, N), R, QL and D-implications.

Motivation

- Although functional inequalities of Modus Ponens and Modus Tollens have quite similar expressions, it is well known that both properties are not equivalent.
- Thus, the simultaneous fulfillment of both properties was studied for the first time for some restricted classes of (S, N), R, QL and D-implications.
- Also, it was proved that when the fuzzy negation N is a strong negation, both properties are equivalent if the fuzzy implication function satisfies the contrapositive symmetry with respect to N.

Following this line of research,

Main objective

We want to analyze which residual implications derived from uninorms, or *RU*-implications for short, satisfy both the Modus Ponens and the Modus Tollens properties with respect to the same t-norm T and a fuzzy negation N (continuous, but not necessarily strong).

Preliminaries

Fuzzy Implication Function

Definition

A binary operator $I : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a fuzzy implication function if it satisfies:

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Definition

A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be a fuzzy implication function if it satisfies:

I1) I is decreasing in the first variable and increasing in the second one,

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Fuzzy Negation

Definition

A function $N : [0,1] \to [0,1]$ is said to be a *fuzzy negation* if it is decreasing with $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is said to be *strong* when it is an involution, i.e.,

$$N(N(x)) = x \text{ for all } x \in [0,1]$$

Negation induced by a t-norm

Let T be a t-norm. A function $N_T : [0,1] \to [0,1]$ defined as

$$N_T(x) = \sup\{y \in [0,1] \mid T(x,y) = 0\},\ x \in [0,1]$$

is called the *natural negation* of T.

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Natural negation

Let I be a fuzzy implication. N_I defined by

$$N_I(x) = I(x,0) \text{ for all } x \in [0,1]$$

is a fuzzy negation, called the *natural negation* of I.

Uninorm

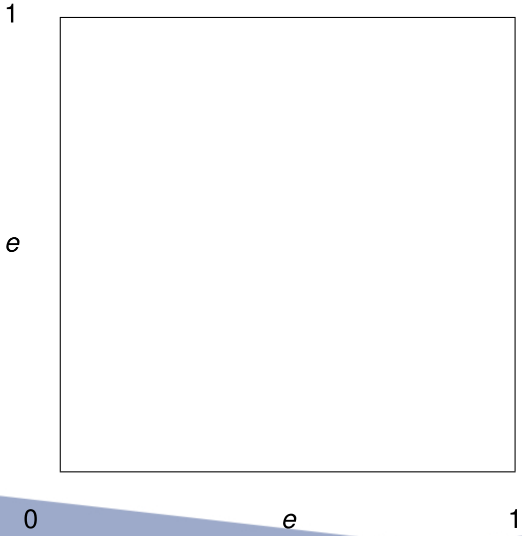
Definition

A *uninorm* is a two-place function $U : [0, 1]^2 \to [0, 1]$ that is associative, commutative, increasing in each place and there exists some element $e \in [0, 1]$, called *neutral element*, such that $U(e, x) = x$ for all $x \in [0, 1]$.

If $e = 0$, U is a t-conorm. If $e = 1$, U is a t-norm.

If $e \in]0, 1[$, U has the following structure:

General structure of a uninorm



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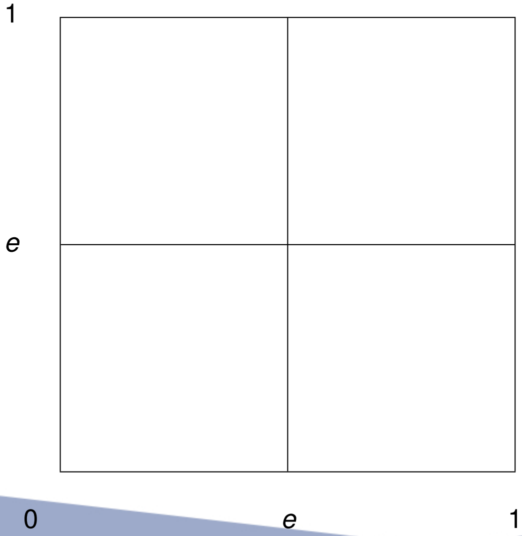

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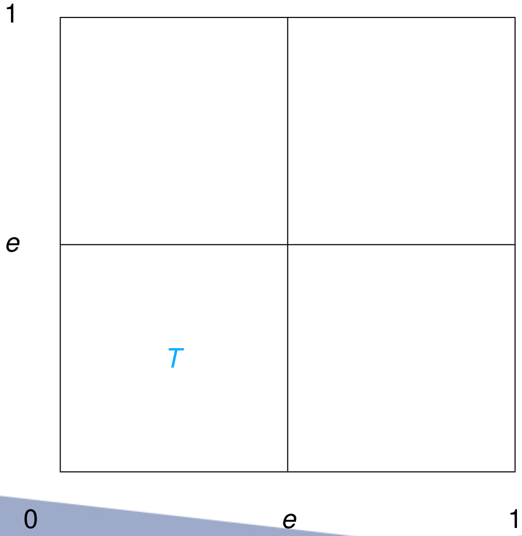

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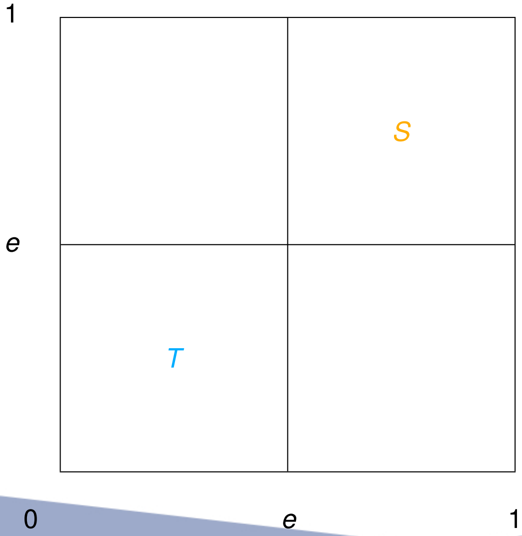

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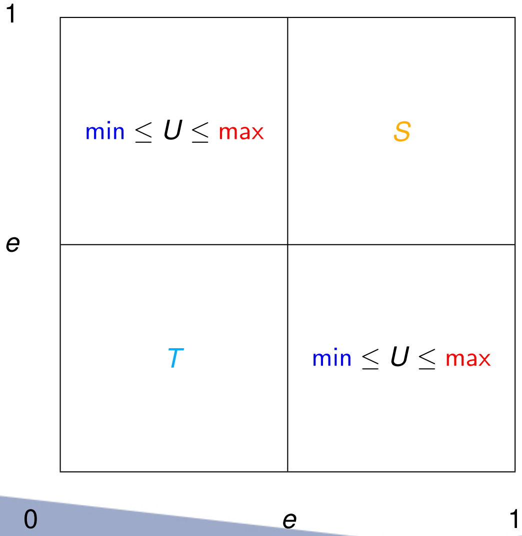
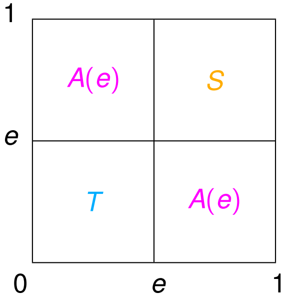


Classes of uninorms

Classes of uninorms



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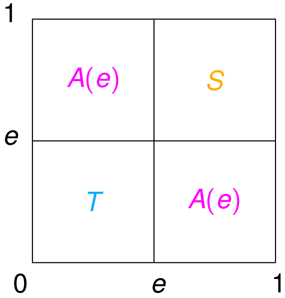

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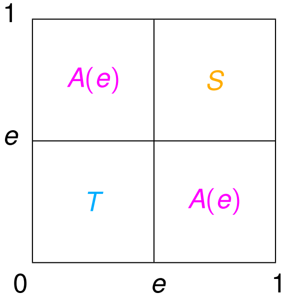

- $\mathcal{U}_{\text{lin}}$, if $U(x, y) \in \{x, y\}$ in $A(e)$

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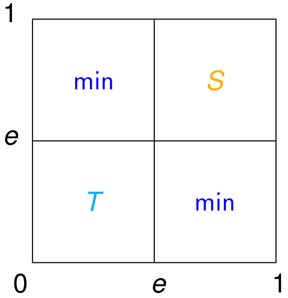

- $\mathcal{U}_{\text{lin}}$, if $U(x, y) \in \{x, y\}$ in $A(e)$
  - $\mathcal{U}_{\min}$, where $U(x, y) = \min(x, y)$ in $A(e)$
    $\langle T, e, S \rangle_{\min}$

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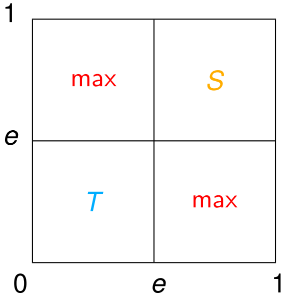

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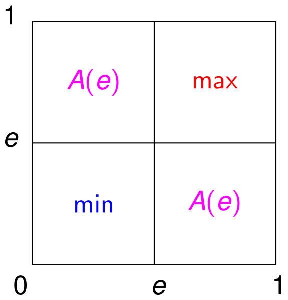

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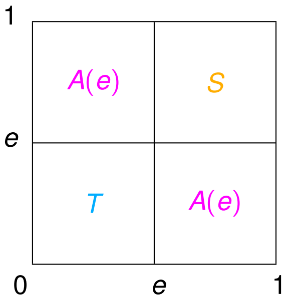
    $\langle T, e, S \rangle_{\max}$
  - $\mathcal{U}_{\text{ide}}$, satisfying $U(x, x) = x$

Classes of uninorms



- $\mathcal{U}_{\text{lin}}$, if $U(x, y) \in \{x, y\}$ in $A(e)$
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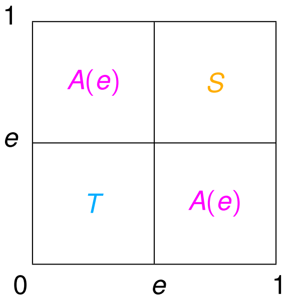

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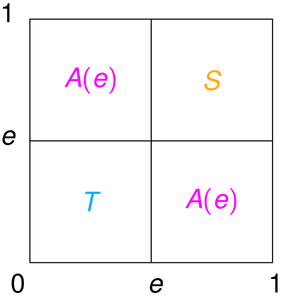
  - $\mathcal{U}_{\cos}$, continuous in $]0, 1[^2$

Classes of uninorms



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  - $\mathcal{U}_{\text{ide}}$, satisfying $U(x, x) = x$
- $\mathcal{U}_{\text{cts}}$, with T i S continuous
  - $\mathcal{U}_{\cos}$, continuous in $]0, 1[^2$
    - $\mathcal{U}_{\text{rep}}$, representable
      $U(x, y) = h^{-1}(h(x) + h(y))$

Idempotent uninorms

Theorem

U is an idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a non increasing function $g : [0,1] \to [0,1]$, symmetric with respect to the identity function, with $g(e) = e$, such that

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)), \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)), \\ x \text{ or } y & \text{if } y = g(x) \text{ and } x = g^2(x), \end{cases}$$

being commutative in the points (x,y) such that $y = g(x)$ with $x = g^2(x)$.

Notation: $U \equiv \langle g, e \rangle_{\mathrm{ide}}$.

Representable uninorms

Definition

A uninorm U, with neutral element $e \in]0,1[$, is called *representable* if there exists a strictly increasing function $h : [0,1] \to [-\infty, +\infty]$ (called an *additive generator* of U, which is unique up to a multiplicative constant $k > 0$), with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$, such that U is given by

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$. We have either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$.

Notation: $U \equiv \langle e, h \rangle_{\text{rep}}$.

Structure of Uninorms continuous in $]0,1[^2$

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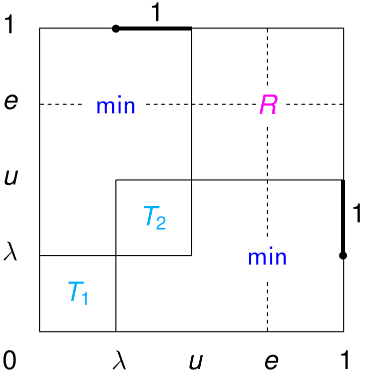
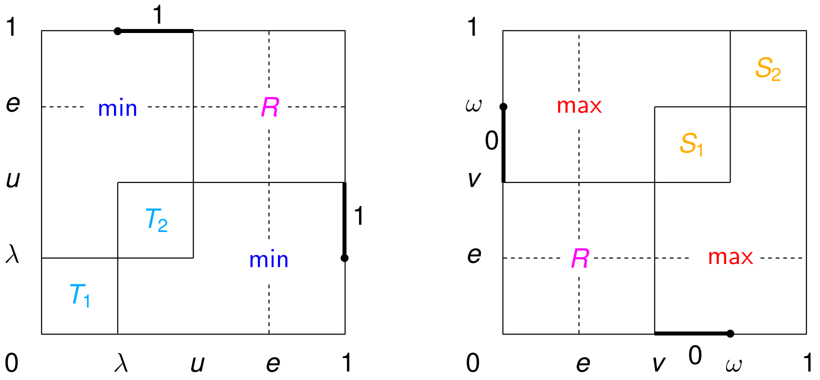


$U_{\cos,\min}$

$$U \equiv \langle T_1, \lambda, T_2, u, (R, e)\rangle_{cos,\min}$$

Structure of Uninorms continuous in $]0,1[^2$



$U_{\cos,\min}$

$U \equiv \langle T_1, \lambda, T_2, u, (R, e)\rangle_{cos,\min}$

$U_{\cos,\max}$

$U \equiv \langle (R, e), v, S_1, \omega, S_2\rangle_{cos,\max}$

RU-implications using uninorms

Definition

Let U be a uninorm. The *residual operation* derived from U is the binary operation given by

$$I_U(x, y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$$

for all $x, y \in [0,1]$.

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for all $x, y \in [0, 1]$.

Proposition

Let U be a uninorm and I_U its residual operation. Then I_U is an implication, called *RU*-implication, if and only if

$$U(x, 0) = 0 \quad \text{for all} \quad x < 1.$$

Modus Ponens Tollens

Definition (Modus Ponens)

Let I be a fuzzy implication function and T a t-norm. It is said that I satisfies the *Modus Ponens property* with respect to T if

$$T(x, I(x, y)) \leq y \text{ for all } x, y \in [0, 1]. \qquad \textbf{(MP)}$$

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Definition (Modus Tollens)

Let I be a fuzzy implication function, T a t-norm and N a fuzzy negation. It is said that I satisfies the *Modus Tollens property* with respect to T and N if

$$T(N(y), I(x, y)) \leq N(x) \text{ for all } x, y \in [0, 1]. \tag{\textbf{MT}}$$

These two properties are not equivalent in general

Example (A fuzzy implication that satisfies (**MP**) but not (**MT**))

Consider $U \equiv \langle h, \frac{3}{4} \rangle_{\text{rep}}$ a representable uninorm with $T_U = T_{\mathbf{P}}$, the product t-norm and S_U any strict t-conorm. Let us consider its residual implication I_U. Let us also consider $T = T_{\mathbf{P}}$ and the negation $N(x) = \frac{1-x}{1+10x}$.

I_U safisfies (**MP**) with respect to T however, I_U does not satisfy (**MT**) with respect to T and N.

These two properties are not equivalent in general

Example (A fuzzy implication that satisfies (**MT**) but not (**MP**))

Let $U \equiv \langle h, \frac{1}{2} \rangle_{\mathrm{rep}}$ be a representable uninorm with additive generator $h(x) = \ln\left(\frac{x}{1-x}\right)$ for all $x \in [0,1]$. Let T be a t-norm whose expression is given by the ordinal sum $T \equiv (\langle 0, \frac{1}{2}, T_{\mathbf{P}} \rangle, \langle \frac{1}{2}, 1, T_1 \rangle)$ with T_1 any continuous t-norm and let us consider the continuous fuzzy negation N given by

$$N(x) = \begin{cases} 1 - x & \text{if } x \leq \frac{1}{2}, \\ \sqrt{x - x^2} & \text{otherwise.} \end{cases}$$

I_U satisfies (**MT**) with respect to T and N however I_U does not satisfy (**MP**) with respect to T.

Modus Ponens Tollens

Definition

Let I be a fuzzy implication function, T a t-norm and N a fuzzy negation. It is said that I satisfies the *Modus Ponens Tollens* (**MPT**) property with respect to T and N whenever equations (**MP**) and (**MT**) are satisfied simultaneously.

A special case that can be considered is when I satisfies the contrapositive symmetry with respect to N.

Definition

Consider I a fuzzy implication function and N a fuzzy negation. Then I satisfies the *contrapositive symmetry* with respect to N if

$$I(x, y) = I(N(y), N(x)) \text{ for all } x, y \in [0, 1]. \tag{CP}$$

Contrapositive symmetry is a well-known property, which is related to the Modus Ponens Tollens as it is stated in the well known result:

Theorem

Consider I a fuzzy implication function, T a t-norm and N a strong negation. If I satisfies the contrapositive symmetry with respect to N, then I satisfies (**MP**) *with respect to T if and only if I satisfies* (**MT**) *with respect to N and T.*

Based on this result if a fuzzy implication function I satisfies (**CP**) with respect to N, only one of (**MP**) or (**MT**) needs to be checked in order to verify that I satisfies (**MPT**).

In the case of residual implications derived from idempotent uninorms, let us recall this result:

Proposition

Consider $U \equiv \langle g, e \rangle_{\mathrm{ide}}$ an idempotent uninorm with $g(0) = 1$, I_U its residual implication and N a strong negation. Then I_U satisfies (**CP**) with respect to N if and only if $g = N$.

As a consequence we have infinite *RU*-implications that satisfy (**MPT**) for any t-norm T, In the case of residual implications derived from idempotent uninorms, let us recall this result:

Proposition

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As a consequence we have infinite *RU*-implications that satisfy (**MPT**) for any t-norm T,

Corollary

Let N be a strong negation, $U \equiv \langle N, e \rangle_{\text{ide}}$ an idempotent uninorm, I_U its residual implication, and T a t-norm. Then I_U satisfies (**MPT**) *with respect to T and N.*

Modus Ponens Tollens for implications derived from different classes of uninorms

Case when U is a uninorm in $\mathcal{U}_{\min}$

A first result,

Proposition

Let $U \equiv \langle T_U, e, S_U \rangle_{\min}$ a uninorm in $\mathcal{U}_{\min}$ and I_U its residual implication. Let T be a continuous t-norm, and N be a continuous fuzzy negation with fixed point $s \in]0,1[$. Then, it holds that:

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- If I_U satisfies (**MPT**) with T and N, then T is nilpotent with normalized additive generator $t : [0,1] \to [0,1]$ and associated negation $N_T(x) = t^{-1}(1 - t(x))$ for all $x \in [0,1]$ such that $N(y) \leq N_T(y)$ for all $y \leq e$.

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Now, let us consider T satisfying the previous conditions. In this case,

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(i) If $T_U = \min$ then I_U always satisfies (**MPT**) with respect to T and N.

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Example

Let us consider the uninorm $U \equiv \langle T_{\mathbf{M}}, e, S_U \rangle_{\min}$ where $T_{\mathbf{M}}$ is the minimum t-norm and S_U any t-conorm. Let $T_{\mathbf{L}}$ be the Łukasiewicz t-norm and $N = N_c$ the classical negation given by $N_c(x) = 1 - x$ for all $x \in [0, 1]$.
Thus, I_U satisfies (**MPT**) with respect to $T_{\mathbf{L}}$ and N_c.

Case when U is a uninorm in $\mathcal{U}_{\min}$

In this case, when

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In this case, when

(ii) If T_U is a strict t-norm with additive generator t_U and either $s \geq e$ or $N(y) = N_T(y)$ for all $y \leq e$, then I_U satisfies (**MPT**) with respect to T and N if and only if the following condition holds:

($\star_1$) Function $g : [0, t(0)] \to [t(e), 1]$ given by the formula $g(x) = t(et_U^{-1}(x))$ is subadditive.

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($\star_1$) Function $g : [0, t(0)] \to [t(e), 1]$ given by the formula $g(x) = t(et_U^{-1}(x))$ is subadditive.

Example

Let us take now $U \equiv \langle T_{\mathbf{P}}, \frac{1}{2}, S_U \rangle_{\min}$ with S_U any t-conorm. Let $T_{\mathbf{L}}$ be the Łukasiewicz t-norm (with additive generator $t(x) = 1 - x$ up to a multiplicative constant), and $N = N_c$. In this example, we are under the conditions of Case (ii) and the function $g : [0, 1] \to [\frac{1}{2}, 1]$ given by $g(x) = 1 - \frac{1}{2}e^{-x}$ is subadditive. Therefore, I_U satisfies (**MPT**) with respect to $T_{\mathbf{L}}$ and N_c.

Case when U is a uninorm in $\mathcal{U}_{\min}$

In this case, when

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(iii) If T_U is a strict t-norm with additive generator t_U, $s < e$ and $N(y) < N_T(y)$ for some $y \leq e$, then I_U satisfies (**MPT**) with respect to T and N if and only if Property $(\star_1)$ is fulfilled and the following condition holds:

$(\star_2)$ For all $y < x < e$,

$$e t_U^{-1}\left(t_U\left(\frac{y}{e}\right) - t_U\left(\frac{x}{e}\right)\right) \leq t^{-1}(t(N(x)) - t(N(y))).$$

Case when U is a uninorm in $\mathcal{U}_{\min}$

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(iv) If T_U is a nilpotent t-norm with additive generator t_U and either $s \geq e$ or $N = N_T$, then I_U satisfies (**MPT**) with respect to T and N if and only if Property $(\star_1)$ and the following property holds:

$(\star_3)$ For all $x \leq e$

$$e \cdot N_{T_U}\left(\frac{x}{e}\right) \leq N_T(x).$$

Case when U is a uninorm in $\mathcal{U}_{\min}$

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(iv) If T_U is a nilpotent t-norm with additive generator t_U and either $s \geq e$ or $N = N_T$, then I_U satisfies (**MPT**) with respect to T and N if and only if Property $(\star_1)$ and the following property holds:

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$$e \cdot N_{T_U}\left(\frac{x}{e}\right) \leq N_T(x).$$

(v) If T_U is a nilpotent t-norm with additive generator t_U, $s < e$ and $N(y) < N_T(y)$ for some $y \leq e$, then I_U satisfies (**MPT**) with respect to T and N if and only if Properties $(\star_1)$, $(\star_2)$ and $(\star_3)$ hold.

Case when U is an Idempotent Uninorm

We know that for an idempotent uninorm $U \equiv \langle g, e \rangle_{\rm ide}$ with $g(0) = 1$, as $T_U = \min$, I_U satisfies (**MP**) with respect to any t-norm. Therefore, we can write the following result.

Proposition

Let $U \equiv \langle g, e \rangle_{ide}$ be an idempotent uninorm with neutral element $e \in]0, 1[$ and such that $g(0) = 1$, T a t-norm and N a fuzzy negation. Then I_U satisfies (**MPT**) with respect to T and N if and only if I_U satisfies (**MT**) with respect to T and N.

Case when U is an Idempotent Uninorm

Now we will distinguish two cases depending on the value of $g(1)$.

Proposition: Case $g(1) > 0$

Let $U \equiv \langle g, e \rangle_{ide}$ with $g(0) = 1$ and $g(1) > 0$ and I_U its residual implication. Let T be a t-norm and N a continuous fuzzy negation. If I_U satisfies the (**MPT**) property with respect to T and N, then the following statements are true:

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(i) $T(N(y), y) = 0$ for all $y \leq g(1)$.

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(i) $T(N(y), y) = 0$ for all $y \leq g(1)$.

(ii) If T is a continuous t-norm then T must be nilpotent with normalized additive generator $t : [0, 1] \to [0, 1]$ and associated negation N_T, which is given by $N_T(x) = t^{-1}(1 - t(x))$, such that $N(y) \leq N_T(y)$ for all $y \leq g(1)$.

Case when U is an Idempotent Uninorm

Theorem (Case $g(1) = 0$)

*Let $U \equiv \langle g, e \rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e \in]0, 1[$ and $g(0) = 1$, $g(1) = 0$ and I_U its residual implication. Let T be a t-norm and N a continuous fuzzy negation. Then I_U satisfies (**MPT**) with respect to T and N if and only if*

$$\min(T(N(y), y), T(N(y), g(x))) \leq N(x) \textit{ for all } y < x.$$

Case when U is an Idempotent Uninorm

Theorem (Case $g(1) = 0$)

*Let $U \equiv \langle g, e \rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e \in]0, 1[$ and $g(0) = 1$, $g(1) = 0$ and I_U its residual implication. Let T be a t-norm and N a continuous fuzzy negation. Then I_U satisfies (**MPT**) with respect to T and N if and only if*

$$\min(T(N(y), y), T(N(y), g(x))) \leq N(x) \textit{ for all } y < x.$$

Example

Let us consider $U \equiv \langle N_c, \frac{1}{2} \rangle_{\text{ide}}$ an idempotent uninorm, $T = T_{\mathbf{L}}$ and $N = N_c$. We have

$$\min(T_{\mathbf{L}}(N_c(y), y), T_{\mathbf{L}}(N_c(y), g(x))) = 0$$

and then I_U satisfies (**MPT**) with respect to $T_{\mathbf{L}}$ and N_c.

Case when U is an Idempotent Uninorm

Proposition(When N is strict)

Let T be a t-norm, N a strict fuzzy negation, and $U \equiv \langle g, e \rangle_{\text{ide}}$ be an idempotent uninorm with neutral element $e \in]0,1[$ with $g(0) = 1$, $g(1) = 0$ and I_U its residual implication. Then I_U satisfies (**MPT**) with respect to T and N if and only if $g(x) \leq N(x)$ for all $x \geq e$.

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Proposition(When N is strict)

Let T be a t-norm, N a strict fuzzy negation, and $U \equiv \langle g, e\rangle_{\mathrm{ide}}$ be an idempotent uninorm with neutral element $e \in]0, 1[$ with $g(0) = 1$, $g(1) = 0$ and I_U its residual implication. Then I_U satisfies (**MPT**) with respect to T and N if and only if $g(x) \leq N(x)$ for all $x \geq e$.

Example

Let us consider $U \equiv \langle g, \frac{1}{4}\rangle_{\mathrm{ide}}$ an idempotent uninorm where

$$g(x) = \begin{cases} 1 - 3x & \text{if } x \leq \frac{1}{3}, \\ 0 & \text{otherwise,} \end{cases}$$

$T = T_{\mathbf{P}}$ and $N = N_c$. It is case we have $g(x) \leq N(x)$ for all $x \geq \frac{1}{4}$. Thus, I_U satisfies (**MPT**) with respect to T and N.

Case when U is a Representable Uninorm

For this kind of uninorms we will consider only continuous t-norms which are not an ordinal sum, namely, the minimum t-norm and continuous Archimedean t-norms.

Proposition

Let $U \equiv \langle h, e \rangle_{\mathrm{rep}}$ be a representable uninorm with neutral element $e \in]0, 1[$ and I_U its residual implication. Let T be a continuous non-ordinal sum t-norm and N a continuous fuzzy negation. Then, it holds that:

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- If I_U satisfies (**MPT**) with T and N, then T is continuous Archimedean with additive generator $t : [0, 1] \to [0, +\infty]$, up to a multiplicative constant.

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(i) I_U satisfies (**MPT**) with respect to T and N if and only if Property $(\star_1)$ is fulfilled and the following property holds:

($\bullet_1$) For all $y \leq x$,

$$h^{-1}(h(y) - h(x)) \leq t^{-1}(t(N(x)) - t(N(y))).$$

Case when U is a Representable Uninorm

Thus, in this case, the following statements are true:

(i) I_U satisfies (**MPT**) with respect to T and N if and only if Property ***($\star_1$)*** is fulfilled and the following property holds:

($\bullet_1$) For all $y \leq x$,

$$h^{-1}(h(y) - h(x)) \leq t^{-1}(t(N(x)) - t(N(y))).$$

(ii) If T is nilpotent and $N = N_T$, I_U satisfies (**MPT**) with respect to T and N if and only if the following property holds:

($\bullet_2$) Function $\phi : [0, 1] \to [-\infty, +\infty]$ given by $\phi(x) = h(t^{-1}(x))$ for all $x \in [0, 1]$ is subadditive.

Case when U is a Representable Uninorm

Example

Let us consider $T = T_{\mathbf{L}}$ and $N = N_c$. Let U be the representable uninorm given by

$$U(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \{(0,0),(1,1)\}, \\ \frac{xy}{xy+(1-x)(1-y)} & \text{otherwise.} \end{cases}$$

which has $e = \frac{1}{2}$ as neutral element and additive generator $h(x) = \ln(\frac{x}{1-x})$. In this case, $\phi(x) = h(t^{-1}(x)) = \ln\left(\frac{1-x}{x}\right)$ which is clearly subadditive. Thus, we conclude that I_U satisfies (**MPT**) with respect to T and N.

Conclusions and Future Work

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- When N is not strong or the contrapositive symmetry is not satisfied, other solutions exist within *RU*-implications derived from uninorms in $\mathcal{U}_{\min}$, representable uninorms and idempotent uninorms.
- For most of these families, necessary and sufficient conditions are presented and in some cases, it is shown that the fulfillment of the Modus Tollens property implies the fulfillment of the Modus Ponens property.

Future Work

We want to complete the results presented in this paper by considering also continuous ordinal sum t-norms as T and to deepen the study in the particular case of idempotent uninorms with $g(0) = 1$ and $g(1) > 0$.

Thank you very much!