Is the invariance with respect to powers of a t-norm a restrictive property? The case of strict t-norms

R. Fernandez-Peralta S. Massanet A. Mir

SCOPIA research group
Dept. Maths and Computer Science
University of the Balearic Islands
Palma, Mallorca

Balearic Islands Health Research Institute (IdISBa)
Palma, Mallorca

http://scopia.uib.eu

Outline

Introduction

Fuzzy implication functions

The definition of fuzzy implication function is enough flexible to allow the existence of a huge number of fuzzy implication functions.

Definition

A binary operation $I : [0,1]^2 \to [0,1]$ is said to be a *fuzzy implication function* if it satisfies:

(I1) $I(x,z) \geq I(y,z)$ when $x \leq y$, for all $z \in [0,1]$.

(I2) $I(x,y) \leq I(x,z)$ when $y \leq z$, for all $x \in [0,1]$.

(I3) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Additional properties

These operators can satisfy additional properties that come usually from tautologies in classical logic.

1. The *identity principle*

$$I(x,x) = 1, \quad x \in [0,1]. \tag{IP}$$

2. The *ordering property*

$$I(x,y) = 1 \Leftrightarrow x \leq y, \quad x,y \in [0,1]. \tag{OP}$$

3. The *exchange principle*

$$I(x, I(y,z)) = I(y, I(x,z)), \quad x,y,z \in [0,1]. \tag{EP}$$

4. The *law of importation* with respect to a t-norm T

$$I(T(x,y),z) = I(x, I(y,z)), \quad x,y,z \in [0,1]. \tag{$\mathbf{LI}_\mathbf{T}$}$$

5. The *left neutrality principle*

$$I(1,y) = y, \quad y \in [0,1]. \tag{NP}$$

6. The *iterative boolean law*

$$I(x,y) = I(x, I(x,y)), \quad x,y \in [0,1]. \tag{IB}$$

Mizumoto and Zimmerman's example

In 1982, Mizumoto and Zimmerman introduced the following example of fuzzy propositions:

If the tomato is red, then it is ripe.
If the tomato is very red, then it is very ripe.
If the tomato is little red, then it is little ripe.

Mizumoto and Zimmerman's example

These fuzzy conditionals involve linguistic modifiers such as *very* or *little* which are usually modeled through Zadeh's potential modifiers:

- very x is computed as x^2,
- little x is computed as $x^{\frac{1}{2}}$.

Although Zadeh used the product t-norm $T_P(x, y) = xy$, any continuous t-norm can be considered to model them.

Powers of continuous t-norms

From a continuous t-norm T, its powers can be defined (C. Walker and E. Walker, 2002). For all $x \in [0,1]$:

- $n \in \mathbb{Z}^+, n \geq 2$: $\quad x_T^{(n)} = T(\overbrace{x, x, \ldots, x}^{n \text{ times}})$.
- $q \in \mathbb{Q}^+$: $\quad x^{\left(\frac{1}{n}\right)} = \sup\{z \in [0,1] \mid z^{(n)} \leq x\}, \quad x^{\left(\frac{m}{n}\right)} = \left(x_T^{\left(\frac{1}{n}\right)}\right)$,
- $r \in \mathbb{R}^+$: $\quad x_T^{(r)} = \lim_{n \to \infty} x_T^{(a_n)}$ where $\lim_{n \to \infty} a_n = r$ with $a_n \in \mathbb{Q}^+$.

Proposition

Let T be a strict t-norm with additive generator t. Then

$$x_T^{(r)} = t^{-1}(rt(x)) \text{ for all } x \in [0,1] \text{ and } r \in [0, +\infty]$$

with the convention that $+\infty \cdot 0 = 0$.

Invariance with respect to T-powers

It is intuitive to think that whenever the same linguistic modifier is applied to both the antecedent and the consequent in Mizumoto and Zimmerman's example, the truth value of the fuzzy conditional should remain the same.

Definition (S. Massanet, J. Recasens and J. Torrens, 2017)

Let I be a fuzzy implication function and T a continuous t-norm. It is said that I is *invariant with respect to T-powers*, or simply that it is *T-power invariant* when

$$I(x, y) = I\left(x_T^{(r)}, y_T^{(r)}\right), \qquad (\mathbf{PI_T})$$

holds for all real number $r > 0$ and for all $x, y \in [0, 1]$ such that $x_T^{(r)}, y_T^{(r)} \neq 0, 1$.

Power-based implications

Definition (S. Massanet, J. Recasens and J. Torrens, 2017)

A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a *T-power based implication* if there exists a continuous t-norm T such that

$$I(x,y) = \sup\{r \in [0,1] \mid y_T^{(r)} \geq x\}, \quad \text{for all } x, y \in [0,1].$$

Let T be a strict t-norm and t an additive generator of T. Then, its power based implication I^T is defined as follows:

$$I^T(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{t(x)}{t(y)} & \text{if } x > y. \end{cases}$$

Characterization of fuzzy implication functions that are invariant with respect to T-powers

Theorem (S. Massanet, J. Recasens and J. Torrens, 2019)

Let T be a strict t-norm and t an additive generator of T. A mapping $I : [0,1]^2 \to [0,1]$ is a fuzzy implication function invariant with respect to T-powers if and only if there exists an increasing mapping $\varphi : [0,+\infty] \to [0,1]$ with $\varphi(0) = 0$, $\varphi(+\infty) = 1$ and such that I is given by

$$I(x,y) = \varphi\left(\frac{t(x)}{t(y)}\right), \quad \text{for all } (x,y) \in [0,1]^2 \setminus \{(x,0),(1,y) | 0 < x,y < 1\}, \quad (1)$$

with the convention that $\frac{0}{0} = \frac{+\infty}{+\infty} = +\infty$, and such that the remaining values $I(x,0)$ and $I(1,y)$ preserve the monotonicity conditions.

Objectives

The aim of our work is to deeply study the family of fuzzy implication functions that are invariant with respect to powers of a strict t-norm T. In this sense, our objectives are:

1. To properly define this family of fuzzy implication functions.
2. To study when do this family fulfill the main additional properties of fuzzy implication functions.
3. To characterize its intersection with the main families of fuzzy implication functions.

Strict T-power invariant implications

Strict T-power invariant implications - Definition

Definition

Let T be a strict t-norm and t an additive generator of T. Let $f : (0,1) \to [0,1]$ be a decreasing function and $\varphi : [0,+\infty] \to [0,1]$, $g : (0,1) \to [0,1]$ increasing functions such that $\varphi(0) = 0$, $\varphi(+\infty) = 1$ and

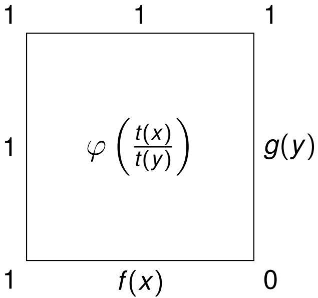
$$\inf_{w \in (0,+\infty)} \varphi(w) \geq \max\left\{ \sup_{y \in (0,1)} g(y), \sup_{x \in (0,1)} f(x) \right\}. \tag{2}$$

The function $I^T_{\varphi,f,g}; [0,1]^2 \to [0,1]$ defined by

$$I^T_{\varphi,f,g}(x,y) = \begin{cases} f(x) & \text{if } x \in (0,1) \text{ and } y = 0, \\ g(y) & \text{if } x = 1 \text{ and } y \in (0,1), \\ \varphi\left(\frac{t(x)}{t(y)}\right) & \text{otherwise}, \end{cases} \tag{3}$$

with the understanding $\frac{0}{0} = \frac{+\infty}{+\infty} = +\infty$, is called a strict T-power invariant implication.

Structure



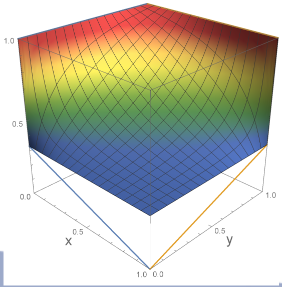
Example 1

Let us consider $t(s) = \frac{1-s}{s}$ for all $s \in [0,1]$, $f(x) = \frac{1-x}{3}$ for all $x \in (0,1)$, $g(y) = \frac{y}{3}$ for all $y \in (0,1)$ and

$$\varphi(w) = \begin{cases} 0 & \text{if } w = 0, \\ \frac{w+1}{w+3} & \text{otherwise.} \end{cases}$$

The corresponding strict T-power invariant implication is given by

$$I_1(x,y) = \begin{cases} 1 & \text{if } (x,y) \in \{(1,1),(0,0)\}, \\ \frac{1-x}{3} & \text{if } x \in (0,1] \text{ and } y = 0, \\ \frac{y}{3} & \text{if } x = 1 \text{ and } y \in (0,1), \\ \frac{y-2xy+x}{y-4xy+3x} & \text{otherwise.} \end{cases}$$



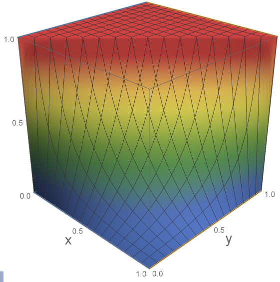
Example 2

Let us consider $t(s) = \frac{1-s}{s}$ for all $s \in [0,1]$, $f(x) = g(y) = 0$ for all $x, y \in (0,1)$ and

$$\varphi(w) = \begin{cases} w & \text{if } w < 1, \\ 1 & \text{otherwise.} \end{cases}$$

The corresponding strict T-power invariant implication is given by

$$I_2(x,y) = \begin{cases} 0 & \text{if } (x \in (0,1] \text{ and } y = 0) \text{ or } (x = 1 \text{ and } y \in (0,1)), \\ \frac{(1-x)y}{(1-y)x} & \text{if } 0 < x < y < 1, \\ 1 & \text{otherwise.} \end{cases}$$

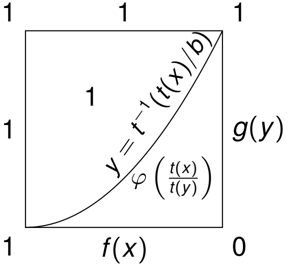


Properties

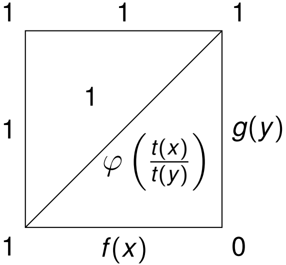
Identity Principle and Ordering Property

Proposition (S. Massanet, J. Recasens and J. Torrens, 2019)

Let $I^T_{\varphi,f,g}$ be a strict T-power invariant implication. Then $I^T_{\varphi,f,g}$ satisfies (**IP**) if and only if $\varphi(1) = 1$. In this case, $I^T_{\varphi,f,g}$ satisfies (**OP**) if and only if $\varphi(w) < 1$ for all $w < 1$.



(i) (**IP**)



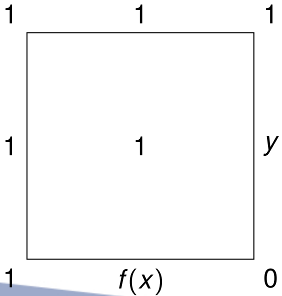
(ii) (**OP**)

Left neutrality Principle

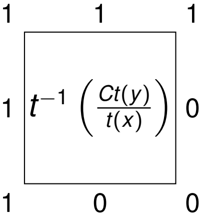
Proposition

Let $I^T_{\varphi,f,g}$ be a strict T-power invariant implication. Then $I^T_{\varphi,f,g}$ satisfies (**NP**) if and only if $g(y) = y$ for all $y \in (0,1)$. Moreover, in this case $I^T_{\varphi,f,g}$ is given by

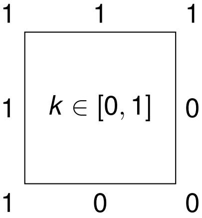
$$I^T_{\varphi,f,g}(x,y) = \begin{cases} f(x) & \text{if } x \in (0,1) \wedge y = 0, \\ y & \text{if } x = 1 \wedge y \in [0,1), \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$



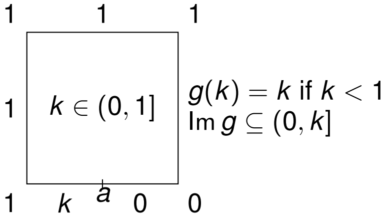
Exchange Principle



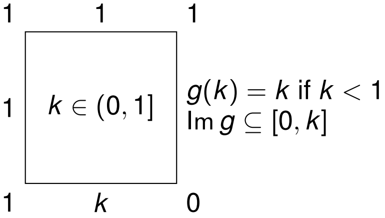
Case (i)



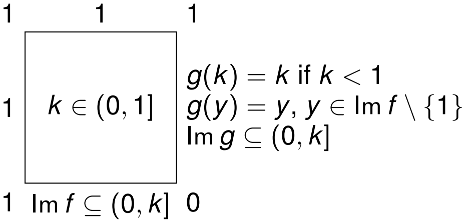
Case (ii)



Case (iii)-(a)



Case (iii)-(b)



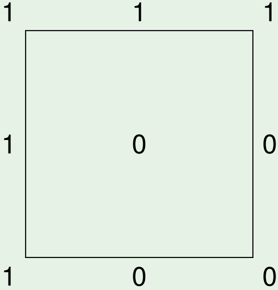
Case (iii)-(c)

Law of importation

Proposition

Let $I^T_{\varphi,f,g}$ be a strict T-power invariant implication and T^* a t-norm. Then $I^T_{\varphi,f,g}$ satisfies (**LI**) with respect to T^* if and only if one of the following conditions hold:

Case 1:

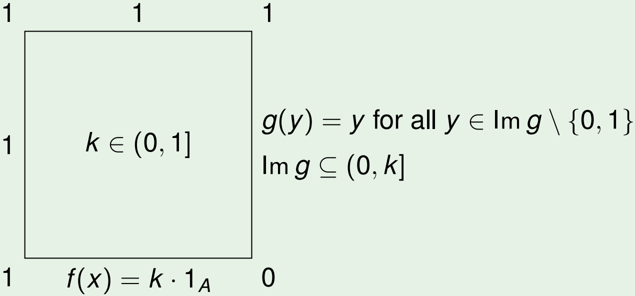


T^* is a positive t-norm.

Law of importation

Proposition (cont.)

<u>Case 2:</u>

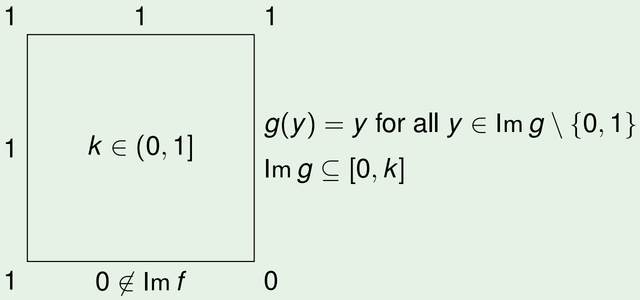


- A is $(0, a)$, $(0, a]$ with $a \in (0, 1)$ or $A = \emptyset$.
- $T^*(x, y) \in (0, 1] \setminus A$ if and only if $x, y \in (0, 1] \setminus A$
- Moreover, if $k < 1$, g must additionally satisfy $g(k) = k$ and T^* must be a positive t-norm.

Law of importation

Proposition (cont.)

Case 3:



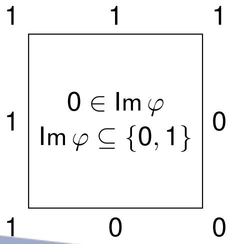
- $f(x) = k$ for all $x \in \operatorname{Im} T^*|_{(0,1)^2} \setminus \{0\}$, $g(y) = y$ for all $y \in \operatorname{Im} f \setminus \{1\}$.
- $g(y) > 0$ for all $y \in (0,1)$ when f is not a function constant to k.
- Moreover, if $k < 1$, g must additionally satisfy $g(k) = k$ and T^* must be a positive t-norm.

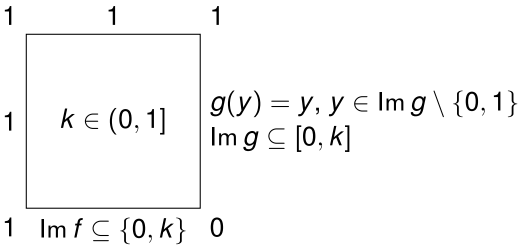
Iterative Boolean Law

Proposition

Let $I^T_{\varphi,f,g}$ be a strict T-power invariant implication. Then $I^T_{\varphi,f,g}$ satisfies (**IB**) if and only if one of the following conditions hold:

1. $\operatorname{Im}\varphi \subseteq \{0,1\}$, φ is not constant to 1 and $f(x) = g(y) = 0$ for all $x, y \in (0,1)$.
2. Let $k \in (0,1]$, then $\varphi(w) = k$ for all $w \in (0,+\infty)$, $\operatorname{Im} f \subseteq \{0,k\}$, $\operatorname{Im} g \subseteq [0,k]$ and $g(y) = y$ for all $y \in \operatorname{Im} g \setminus \{0,1\}$.

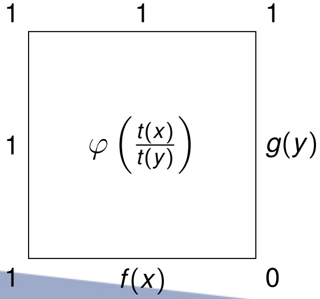




Is the invariance with respect to T-powers a restrictive property?

Remark

According to the previous study, all strict T-power invariant implications that satisfy either (**NP**), (**EP**), (**IB**) and (**LI**) except two cases, are given by a φ which is constant in $(0, +\infty)$.
These fuzzy implication functions are given by an expression which is independent from the generator of the corresponding t-norm. Therefore, they are invariant with respect to T-powers of any strict t-norm!



Intersections

Intersections

Let us denote the family of strict T-power implication functions as:

$$\mathbb{I}^{T}_{\varphi,f,g} \quad - \quad \text{the family of all strict } T\text{-power invariant implications.}$$

First of all, we consider the following well-known families of fuzzy implication functions:

$$\begin{aligned}
\mathbb{I}_{\mathbb{S},\mathbb{N}} \quad &- \quad \text{the family of all } (S, N)\text{-implications;} \\
\mathbb{I}_{\mathbb{T}} \quad &- \quad \text{the family of all } R\text{-implications;} \\
\mathbb{I}_{\mathbb{QL}} \quad &- \quad \text{the family of all } QL\text{-implications;} \\
\mathbb{I}_{\mathbb{D}} \quad &- \quad \text{the family of all } D\text{-implications;} \\
\mathbb{I}_{\mathbb{F}} \quad &- \quad \text{the family of all } f\text{-generated implications;} \\
\mathbb{I}_{\mathbb{G}} \quad &- \quad \text{the family of all } g\text{-generated implications;} \\
\mathbb{I}_{\mathbb{H}} \quad &- \quad \text{the family of all } h\text{-generated implications.}
\end{aligned}$$

Intersections

Proposition

1. $\mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{T}} = I_{\mathbf{WB}}$.
2. $\mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{S},\mathbb{N}} = \mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{QL}} = \mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{D}} = I_1$.
3. $\mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{F}} = \mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{G}} = \mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{H}} = \emptyset$.

$$I_{\mathbf{WB}}(x,y) = \begin{cases} y & \text{if } x = 1 \wedge y \in [0,1], \\ 1 & \text{otherwise,} \end{cases}$$

$$I_1(x,y) = \begin{cases} f(x) & \text{if } x \in (0,1) \wedge y = 0, \\ y & \text{if } x = 1 \wedge y \in [0,1), \\ 1 & \text{otherwise,} \end{cases}$$

where $f : (0,1) \to [0,1]$ is a decreasing function with $\operatorname{Im} f \subseteq (0,1]$

Intersections

Let us now consider some of the most well-known families that do not satisfy (**NP**):

$$\begin{aligned} \mathbb{I}_{\mathbb{U},\mathbb{N}} &\quad - \quad \text{the family of all } (U, N)\text{-implications;} \\ \mathbb{I}_{\mathbb{U}} &\quad - \quad \text{the family of all } RU\text{-implications;} \\ \mathbb{I}_{\mathbb{H},e} &\quad - \quad \text{the family of all } (h, e)\text{-implications.} \end{aligned}$$

Intersections

Proposition

1. $\{I_{Lt}, I_1, I_2, I_3\} \subset \mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{U},\mathbb{N}}$.
2. $\mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{U}} = I_3$.
3. $\mathbb{I}^{T}_{\varphi,f,g} \cap \mathbb{I}_{\mathbb{H},e} = \emptyset$.

$$I_2(x,y) = \begin{cases} 0 & \text{if } x = 1 \wedge y = 0, \\ g(y) & \text{if } x = 1 \wedge y \in (0,1), \\ 1 & \text{otherwise}, \end{cases}$$

$$I_3(x,y) = \begin{cases} 0 & \text{if } (x \in (0,1] \wedge y = 0) \vee (x = 1 \wedge y \in (0,1)), \\ t^{-1}\left(\frac{Ct(y)}{t(x)}\right) & \text{otherwise}, \end{cases}$$

where $C \in (0,+\infty)$, $g : (0,1) \to [0,1]$ is an increasing function with $g(y) \in \{0, y\}$ for all $y \in (0,1)$ and t is an additive generator of a strict t-norm.

Conclusions and future work

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In this work we have focused on the family of fuzzy implication functions characterized by the fact that they fulfill the invariance with respect to powers of a strict t-norm.

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- We have studied when do this family satisfy another additional property:
  - There exist available examples of fuzzy implication functions which satisfy the invariance property with respect to powers of a strict t-norm and the corresponding additional property.

Conclusions

In this work we have focused on the family of fuzzy implication functions characterized by the fact that they fulfill the invariance with respect to powers of a strict t-norm.

- We have studied when do this family satisfy another additional property:
  - There exist available examples of fuzzy implication functions which satisfy the invariance property with respect to powers of a strict t-norm and the corresponding additional property.
  - The invariance with respect to powers of a strict t-norm is a quite restrictive property.
- We have completely characterized the intersection with almost all of the main families of fuzzy implication functions.

Future work

As future work, we want to perform a similar study for the family of nilpotent T-power invariant implications and also for those fuzzy implication functions which are invariant with respect to powers of an ordinal sum t-norm.

Thank you for your attention!